First-Order Logic

Part Two

Recap from Last Time

What is First-Order Logic?

- *First-order logic* is a logical system for reasoning about properties of objects.

- Augments the logical connectives from propositional logic with

  - *predicates* that describe properties of objects,

  - *functions* that map objects to one another, and

  - *quantifiers* that allow us to reason about many objects at once.

Some rabbit is cute.

$$\exists m. \ (Rabbit(m) \land Cute(m))$$

∃ is the **existential quantifier** and says "for some choice of m, the following is true."

"For any natural number n,
n is even if and only if n^2 is even"

$\forall n.$ $(n \in \mathbb{N} \rightarrow (Even(n) \leftrightarrow Even(n^2)))$

∀ is the **universal quantifier**
and says "for any choice of n,
the following is true."

"Some *P* is a *Q*"

translates as

$$\exists x.\ (P(x) \wedge Q(x))$$

Useful Intuition:

Existentially-quantified statements are false unless there's a positive example.

$$\exists x.\ (P(x) \wedge Q(x))$$

If x is an example, it *must* have property P on top of property Q.

"All P's are Q's"

translates as

$$\forall x.\ (P(x) \rightarrow Q(x))$$

Useful Intuition:

Universally-quantified statements are true unless there's a counterexample.

$$\forall x. \ (P(x) \rightarrow Q(x))$$

If x is a counterexample, it *must* have property *P* but not have property *Q*.

New Stuff!

The Aristotelian Forms

"All As are Bs"

$\forall x.\ (A(x) \to B(x))$

"Some As are Bs"

$\exists x.\ (A(x) \wedge B(x))$

"No As are Bs"

$\forall x.\ (A(x) \to \neg B(x))$

"Some As aren't Bs"

$\exists x.\ (A(x) \wedge \neg B(x))$

It is worth committing these patterns to memory. We'll be using them throughout the day and they form the backbone of many first-order logic translations.

The Art of Translation

Using the predicates

- *Person*(*p*), which states that *p* is a person, and
- *Loves*(*x*, *y*), which states that *x* loves *y*,

write a sentence in first-order logic that means "every person loves someone else."

Every person loves someone else

Every person loves some other person

Every person p loves some other person

Every person p loves some other person

> "All *A*s are *B*s"
>
> $\forall x.\ (A(x) \rightarrow B(x))$

$\forall p.\ (Person(p) \rightarrow$

   p loves some other person

$)$

"All As are Bs"

$\forall x.\ (A(x) \rightarrow B(x))$

$\forall p. \ (Person(p) \rightarrow$

    p loves some other person

)

$\forall p. \ (Person(p) \rightarrow$
  there is some other person that p loves

)

$\forall p. \ (Person(p) \rightarrow$
  there is a person other than p that p loves

)

$\forall p.\ (Person(p) \rightarrow$

   there is a person q, other than p, where p loves q

$)$

$\forall p.\ (Person(p) \rightarrow$

there is a person q, other than p, where
p loves q

$)$

$\forall p. (Person(p) \rightarrow$

there is a person q, other than p, where
p loves q

$)$

"Some As are Bs"

$\exists x. (A(x) \wedge B(x))$

$\forall p.\ (Person(p) \rightarrow$
    $\exists q.\ (Person(q) \land$ *, other than p, where*
        p loves q
    $)$
$)$

"Some *A*s are *B*s"

$\exists x.\ (A(x) \land B(x))$

$\forall p.\ (Person(p) \rightarrow$
$\quad \exists q.\ (Person(q) \land$, *other than p, where*
$\qquad$ *p loves q*
$\quad)$
$)$

$\forall p.\ (Person(p) \to$
$\quad \exists q.\ (Person(q) \land p \neq q \land$
$\qquad$ *p loves q*
$\quad)$
$)$

$\forall p.\ (Person(p) \rightarrow$
$\quad \exists q.\ (Person(q) \land p \neq q \land$
$\qquad Loves(p, q)$
$\quad)$
$)$

Using the predicates

   - *Person*(*p*), which states that *p* is a person, and
   - *Loves*(*x*, *y*), which states that *x* loves *y*,

write a sentence in first-order logic that means "there is a person that everyone else loves."

There is a person that everyone else loves

There is a person p where everyone else loves p

There is a person p where everyone else loves p

"Some *A*s are *B*s"

$\exists x. (A(x) \wedge B(x))$

$\exists p. \ (Person(p) \ \wedge$

everyone else loves p

$)$

"Some *A*s are *B*s"

$\exists x. \ (A(x) \ \wedge \ B(x))$

$\exists p.\ (Person(p)\ \wedge$
    everyone else loves p

$)$

$\exists p.\ (Person(p) \wedge$

every other person q loves p

$)$

$\exists p. \ (Person(p) \ \wedge$
    every person q, other than p, loves p

$)$

$\exists p.\ (Person(p) \wedge$

every person q, other than p, loves p

$)$

> "All *A*s are *B*s"
>
> $\forall x.\ (A(x) \rightarrow B(x))$

$\exists p.$ *(Person(p)* ∧

   $\forall q.$ *(Person(q)* ∧ *p* ≠ *q* →

      q loves p

)

)

"All *A*s are *B*s"

$\forall x.\ (A(x) \rightarrow B(x))$

$\exists p.\ (Person(p) \wedge$
$\quad \forall q.\ (Person(q) \wedge p \neq q \rightarrow$
$\qquad$ *q loves p*
$\quad)$
$)$

$\exists p.\ (Person(p) \land$

$\quad \forall q.\ (Person(q) \land p \neq q \rightarrow$

$\qquad Loves(q,\ p)$

$\quad)$

$)$

Combining Quantifiers

- Most interesting statements in first-order logic require a combination of quantifiers.

- Example: "Every person loves someone else"

For every person… $\forall p. (Person(p) \rightarrow$

… there is another person … $\exists q. (Person(q) \wedge p \neq q \wedge$

… they love $Loves(p, q)$

$)$

$)$

Combining Quantifiers

- Most interesting statements in first-order logic require a combination of quantifiers.

- Example: "There is someone everyone else loves."

There is a person…

… that everyone else …

… loves.

$\exists p. (Person(p) \land$
$\quad \forall q. (Person(q) \land p \neq q \rightarrow$
$\qquad Loves(q, p)$
$\quad)$
$)$

For Comparison

For every person…
$\forall p. \; (Person(p) \rightarrow$

… there is another person …
$\exists q. \; (Person(q) \land p \neq q \; \land$

… they love
$Loves(p, q)$

$)$

$)$

There is a person…
$\exists p. \; (Person(p) \; \land$

… that everyone else …
$\forall q. \; (Person(q) \land p \neq q \rightarrow$

… loves.
$Loves(q, p)$

$)$

$)$

Every Person Loves Someone Else
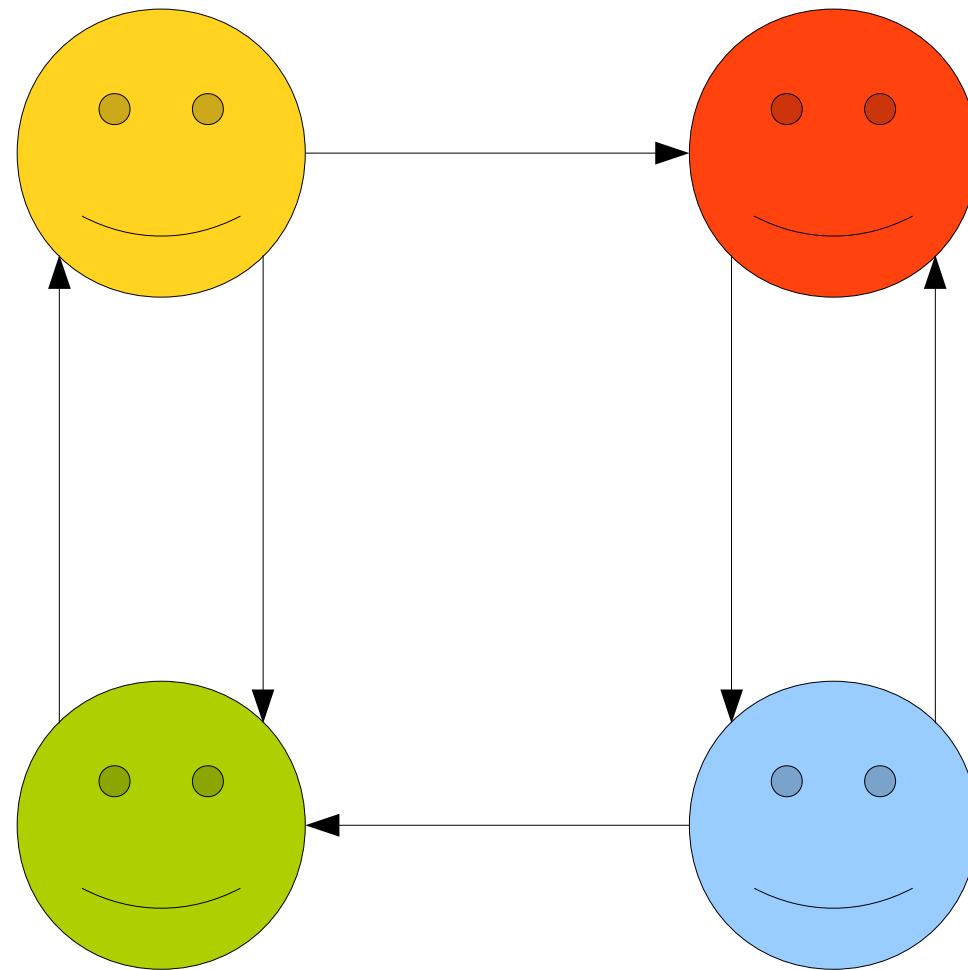
Every Person Loves Someone Else
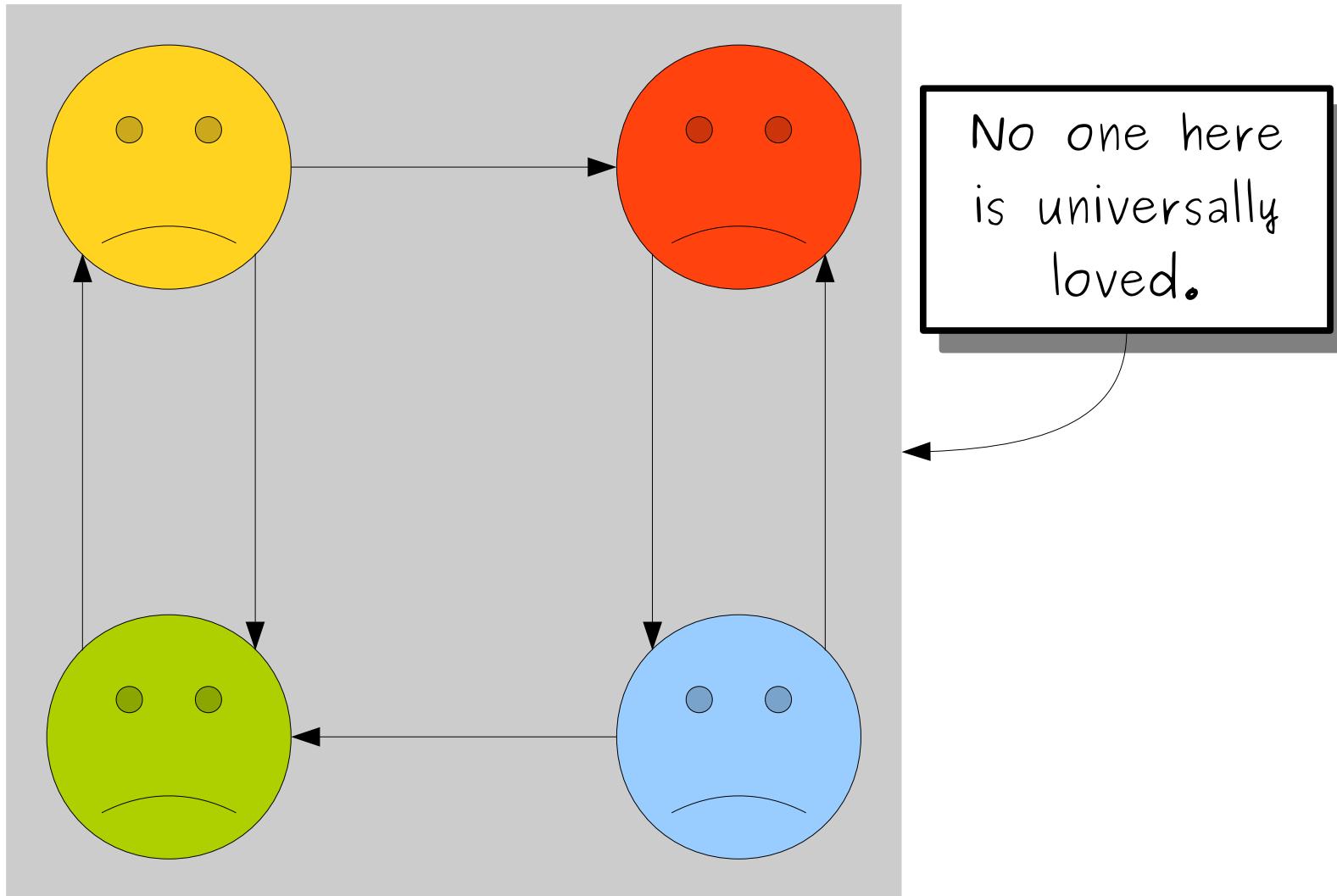
There is Someone Everyone Else Loves

There is Someone Everyone Else Loves

This person does not love anyone else.
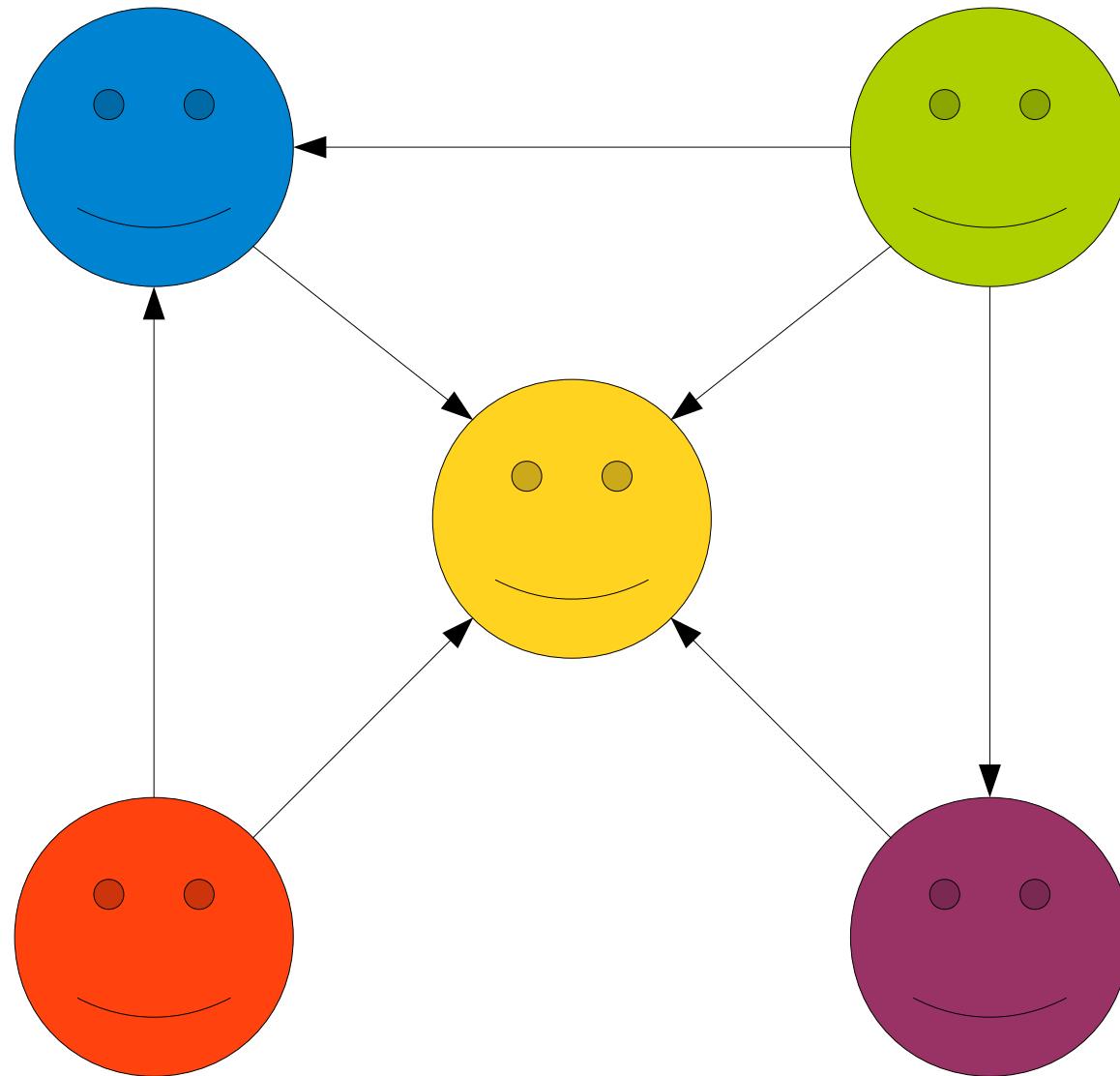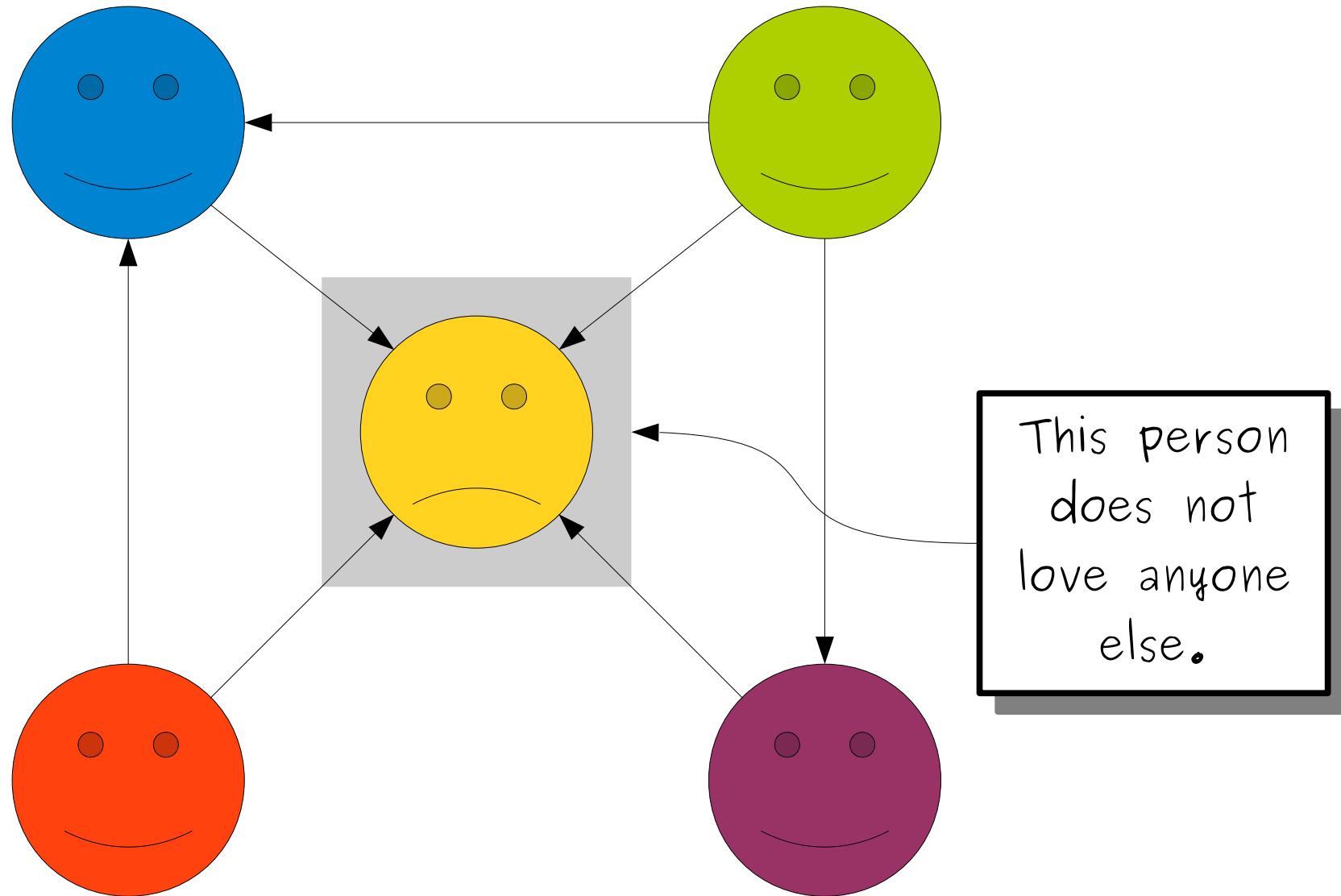
Every Person Loves Someone Else *and* There is Someone Everyone Else Loves
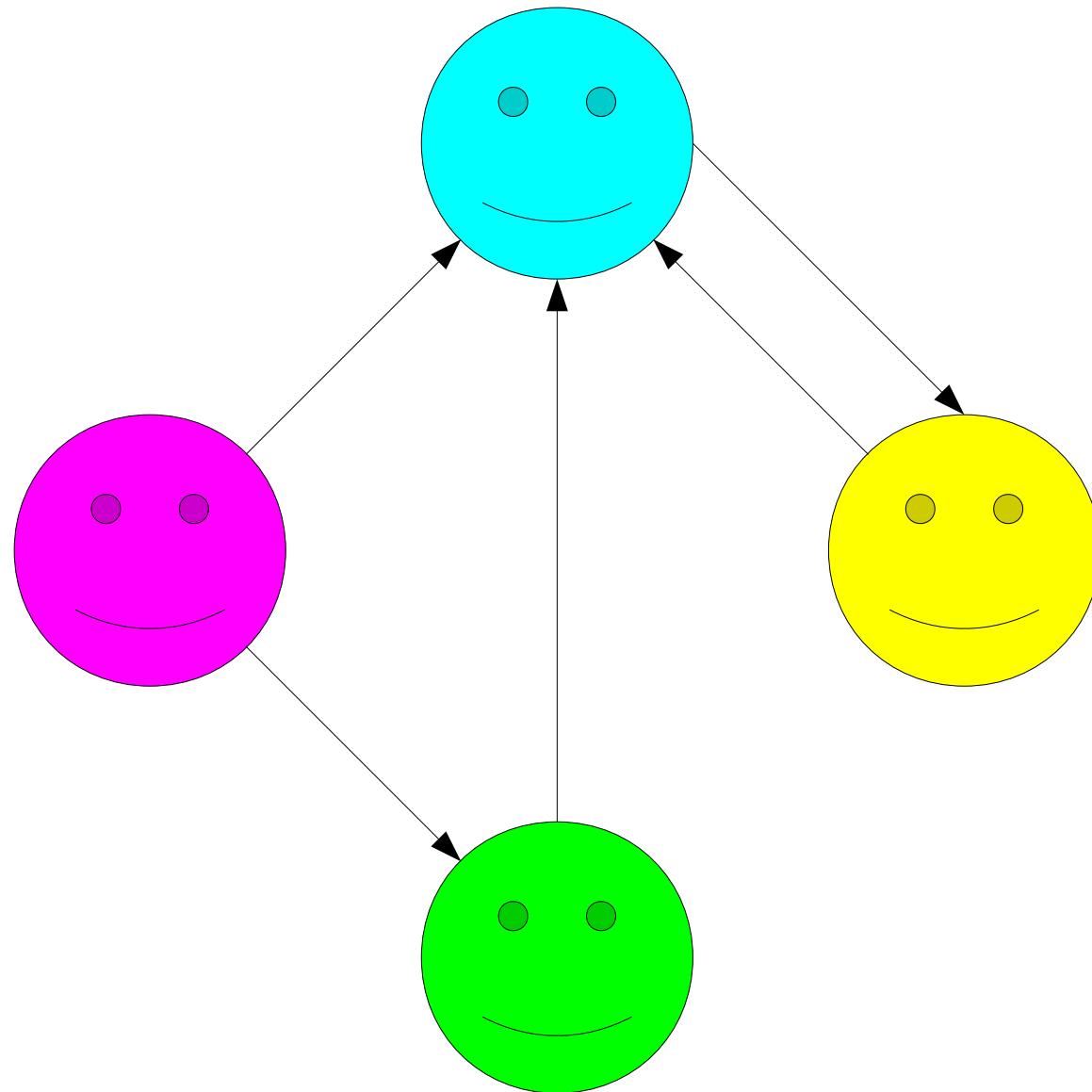
For every person… $\forall p.\ (Person(p) \rightarrow$

… there is another person … $\exists q.\ (Person(q) \land p \neq q\ \land$

… they love $Loves(p,\ q)$

$)$

$)$

and $\land$

There is a person… $\exists p.\ (Person(p)\ \land$

… that everyone else … $\forall q.\ (Person(q) \land p \neq q \rightarrow$

… loves. $Loves(q,\ p)$

$)$

$)$

Restricted Quantifiers

Quantifying Over Sets

- The notation

$$\forall x \in S.\ P(x)$$

  means "for any element x of set S, $P(x)$ holds." (It's vacuously true if S is empty.)

- The notation

$$\exists x \in S.\ P(x)$$

  means "there is an element x of set S where $P(x)$ holds." (It's false if S is empty.)

Quantifying Over Sets

- The syntax

$$\forall x \in S. \, P(x)$$

$$\exists x \in S. \, P(x)$$

  is allowed for quantifying over sets.

- In CS103, feel free to use these restricted quantifiers, but please do not use variants of this syntax.

- For example, don't do things like this:

$$\forall x \text{ with } P(x). \, Q(x)$$

$$\forall y \text{ such that } P(y) \wedge Q(y). \, R(y).$$

$$\exists P(x). \, Q(x)$$

Quantifier Ordering

Quantifier Ordering

- Consider these two first-order formulas:

$$\forall m \in \mathbb{N}.\ \exists n \in \mathbb{N}.\ m < n.$$

$$\exists n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ m < n.$$

- One of these statements is true. One is false. Which is which?

- Why?

Quantifier Ordering

- Consider these two first-order formulas:

$$\forall m \in \mathbb{N}.\ \exists n \in \mathbb{N}.\ m < n.$$

$$\exists n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ m < n.$$

- This says

**for every natural number m,
there's a larger natural number n.**

- This is true: given any $m \in \mathbb{N}$, we can choose n to be $m + 1$.

- Notice that we can pick n based on m, and we don't have to pick the same n each time.

Quantifier Ordering

- Consider these two first-order formulas:

$$\forall m \in \mathbb{N}.\ \exists n \in \mathbb{N}.\ m < n.$$

$$\exists n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ m < n.$$

- This says

there is a natural number n
that's larger than every $m \in \mathbb{N}$

- This is false: no natural number is bigger than every natural number.

- Because $\exists n \in \mathbb{N}$ comes first, we have to make a single choice of n that works.

Quantifier Ordering

- The statement

$$\forall x.\ \exists y.\ P(x, y)$$

means "for any choice of x, there's some choice of y where $P(x, y)$ is true."

- The choice of y can be different every time and can depend on x.

Quantifier Ordering

- The statement

$$\exists x. \, \forall y. \, P(x, y)$$

means "there is some x where for any choice of y, we get that $P(x, y)$ is true."

- Since the inner part has to work for any choice of y, this places a lot of constraints on what x can be.

Order matters when mixing existential and universal quantifiers!

Set Translations

Using the predicates

   - $Set(S)$, which states that S is a set, and
   - $x \in y$, which states that x is an element of y,

write a sentence in first-order logic that means "the empty set exists."

Using the predicates

    - $Set(S)$, which states that S is a set, and
    - $x \in y$, which states that x is an element of y,

write a sentence in first-order logic that means "the empty set exists."

First-order logic doesn't have set operators or symbols "built in." If we only have the predicates given above, how might we describe this?

The empty set exists.

There is some set S that is empty.

$\exists S.\ (Set(S)\ \wedge$
   S is empty.
)

$\exists S. \ (Set(S) \ \wedge$

   there are no elements in S

)

$\exists S.\ (Set(S)\ \wedge$

    ¬there is an element in S

)

$\exists S.\ (Set(S)\ \land$
  $\neg$*there is an element x in S*
)

$$\exists S. \ (Set(S) \ \land$$
$$\neg \exists x. \ x \in S$$
$$)$$

$$\exists S.\ (Set(S) \land \neg \exists x.\ x \in S)$$

$\exists S. \ (Set(S) \ \wedge \ \neg\exists x. \ x \in S)$

$\exists S. \ (Set(S) \ \wedge$
   there are no elements in S
)

$\exists S. \, (Set(S) \, \land \, \neg\exists x. \, x \in S)$

$\exists S. \, (Set(S) \, \land$
   every object does not belong to S
$)$

$\exists S. \ (Set(S) \ \wedge \ \neg \exists x. \ x \in S)$

$\exists S. \ (Set(S) \ \wedge$
   every object x does not belong to S
)

$\exists S. \ (Set(S) \land \neg \exists x. \ x \in S)$

$\exists S. \ (Set(S) \land$
   $\forall x. \ x \notin S$
$)$

$$\exists S. \ (Set(S) \land \neg \exists x. \ x \in S)$$

$$\exists S. \ (Set(S) \land \forall x. \ x \notin S)$$

$\exists S. \ (Set(S) \land \neg \exists x. \ x \in S)$

$\exists S. \ (Set(S) \land \forall x. \ x \notin S)$

Both of these translations are correct. Just like in propositional logic, there are many different equivalent ways of expressing the same statement in first-order logic.

$$\exists S. \; (Set(S) \; \wedge \; \neg \exists x. \; x \in S)$$

$$\exists S. \; (Set(S) \; \wedge \; \forall x. \; x \notin S)$$

$\exists S.\ (Set(S) \land \neg\exists x.\ x \in S)$

$\exists S.\ (Set(S) \land \forall x.\ x \notin S)$

Why can we switch which quantifier we're using here?

Mechanics: Negating Statements

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all objects x, $P(x)$ is true.	There is an x where $P(x)$ is false.
$\exists x.\ P(x)$	There is an x where $P(x)$ is true.	For all objects x, $P(x)$ is false.
$\forall x.\ \neg P(x)$	For all objects x, $P(x)$ is false.	There is an x where $P(x)$ is true.
$\exists x.\ \neg P(x)$	There is an x where $P(x)$ is false.	For all objects x, $P(x)$ is true.

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	When is this true?	When is this false?
$\forall x.\ P(x)$	For all objects x, $P(x)$ is true.	**There is an x where $P(x)$ is false.**
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$\forall x.\ \neg P(x)$	For all objects x, $P(x)$ is false.	There is an x where $P(x)$ is true.
$\exists x.\ \neg P(x)$	**There is an x where $P(x)$ is false.**	For all objects x, $P(x)$ is true.

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$\forall x.\ \neg P(x)$	For all objects x, $P(x)$ is false.	There is an x where $P(x)$ is true.
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$\forall x.\ \neg P(x)$	For all objects x, $P(x)$ is false.	There is an x where $P(x)$ is true.
$\exists x.\ \neg P(x)$	There is an x where $P(x)$ is false.	For all objects x, $P(x)$ is true.

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all objects x, $P(x)$ is true.	$\exists x.\ \neg P(x)$
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$\exists x.\ \neg P(x)$	There is an x where $P(x)$ is false.	For all objects x, $P(x)$ is true.

An Extremely Important Table

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$\exists x.\ P(x)$	There is an x where $P(x)$ is true.	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	**For all objects x, $P(x)$ is false.**	There is an x where $P(x)$ is true.
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An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all objects x, $P(x)$ is true.	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	There is an x where $P(x)$ is true.	$\forall x.\ \neg P(x)$
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An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all objects x, $P(x)$ is true.	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	**There is an x where $P(x)$ is true.**	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all objects x, $P(x)$ is false.	**There is an x where $P(x)$ is true.**
$\exists x.\ \neg P(x)$	There is an x where $P(x)$ is false.	For all objects x, $P(x)$ is true.

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all objects x, $P(x)$ is true.	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	**There is an x where $P(x)$ is true.**	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all objects x, $P(x)$ is false.	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	There is an x where $P(x)$ is false.	For all objects x, $P(x)$ is true.

An Extremely Important Table

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$\forall x.\ \neg P(x)$	For all objects x, $P(x)$ is false.	$\exists x.\ P(x)$
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An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	**For all objects x, $P(x)$ is true.**	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	There is an x where $P(x)$ is true.	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all objects x, $P(x)$ is false.	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	There is an x where $P(x)$ is false.	**For all objects x, $P(x)$ is true.**

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	**For all objects x, $P(x)$ is true.**	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	There is an x where $P(x)$ is true.	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all objects x, $P(x)$ is false.	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	There is an x where $P(x)$ is false.	$\forall x.\ P(x)$

An Extremely Important Table

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$\forall x.\ P(x)$	For all objects x, $P(x)$ is true.	$\exists x.\ \neg P(x)$
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$\exists x.\ \neg P(x)$	There is an x where $P(x)$ is false.	$\forall x.\ P(x)$

Negating First-Order Statements

- Use the equivalences

$$\neg \forall x.\ A \quad \textit{is equivalent to} \quad \exists x.\ \neg A$$

$$\neg \exists x.\ A \quad \textit{is equivalent to} \quad \forall x.\ \neg A$$

to negate quantifiers.

- Mechanically:

  - Push the negation across the quantifier.
  - Change the quantifier from $\forall$ to $\exists$ or vice-versa.

- Use techniques from propositional logic to negate connectives.

Taking a Negation

$$\forall x. \, \exists y. \, Loves(x, y)$$

("Everyone loves someone.")

$$\neg \forall x. \, \exists y. \, Loves(x, y)$$
$$\exists x. \, \neg \exists y. \, Loves(x, y)$$
$$\exists x. \, \forall y. \, \neg Loves(x, y)$$

("There's someone who doesn't love anyone.")

Two Useful Equivalences

- The following equivalences are useful when negating statements in first-order logic:

$$\neg(p \wedge q) \quad \textbf{\textit{is equivalent to}} \quad p \rightarrow \neg q$$

$$\neg(p \rightarrow q) \quad \textbf{\textit{is equivalent to}} \quad p \wedge \neg q$$

- These identities are useful when negating statements involving quantifiers.

  - $\wedge$ is used in existentially-quantified statements.

  - $\rightarrow$ is used in universally-quantified statements.

- When pushing negations across quantifiers, we *strongly recommend* using the above equivalences to keep $\rightarrow$ with $\forall$ and $\wedge$ with $\exists$.

Negating Quantifiers

- What is the negation of the following statement, which says "there is a cute puppy"?

$$\exists x.\ (Puppy(x) \land Cute(x))$$

- We can obtain it as follows:

$$\neg\exists x.\ (Puppy(x) \land Cute(x))$$

$$\forall x.\ \neg(Puppy(x) \land Cute(x))$$

$$\forall x.\ (Puppy(x) \rightarrow \neg Cute(x))$$

- This says "no puppy is cute."

- Do you see why this is the negation of the original statement from both an intuitive and formal perspective?

$$\exists S.\ (Set(S) \wedge \forall x.\ \neg(x \in S))$$

("There is a set with no elements.")

$$\neg \exists S.\ (Set(S) \wedge \forall x.\ \neg(x \in S))$$
$$\forall S.\ \neg(Set(S) \wedge \forall x.\ \neg(x \in S))$$
$$\forall S.\ (Set(S) \rightarrow \neg \forall x.\ \neg(x \in S))$$
$$\forall S.\ (Set(S) \rightarrow \exists x.\ \neg\neg(x \in S))$$
$$\forall S.\ (Set(S) \rightarrow \exists x.\ x \in S)$$

("Every set contains at least one element.")

Expressing Uniqueness

Using the predicate

  - *WayToFindOut(w)*, which states that *w* is a way to find out,

write a sentence in first-order logic that means "there is only one way to find out."

There is only one way to find out.

Something is a way to find out, and nothing else is.

Some thing w is a way to find out, and nothing else is.

Some thing w is a way to find out, and nothing besides w is a way to find out

$\exists w. \, (WayToFindOut(w) \land$

nothing besides w is way to find out

$)$

$\exists w. \ (WayToFindOut(w) \ \wedge$
 anything that isn't w isn't a way to find out
$)$

$\exists w.\ (WayToFindOut(w) \land$

   any thing x that isn't w isn't a way to find out
)

$\exists w. \ (WayToFindOut(w) \ \wedge$
    $\forall x. \ (x \neq w \rightarrow$ *x isn't a way to find out*$)$
$)$

$\exists w.\ (WayToFindOut(w) \land$
$\quad \forall x.\ (x \neq w \to \neg WayToFindOut(x))$
$)$

$\exists w. \ (WayToFindOut(w) \ \land$
$\qquad \forall x. \ (x \neq w \rightarrow \neg WayToFindOut(x))$
$)$

$$\exists w. \, (WayToFindOut(w) \; \wedge$$
$$\quad \forall x. \, ({\color{blue} WayToFindOut(x) \rightarrow x = w})$$
$$)$$

$$\exists w. \ (WayToFindOut(w) \ \wedge$$
$$\quad \forall x. \ (WayToFindOut(x) \rightarrow x = w)$$
$$)$$

Expressing Uniqueness

- To express the idea that there is exactly one object with some property, we write that

  - there exists at least one object with that property, and that

  - there are no other objects with that property.

- You sometimes see a special "uniqueness quantifier" used to express this:

$$\exists !x.\ P(x)$$

- For the purposes of CS103, please do not use this quantifier. We want to give you more practice using the regular $\forall$ and $\exists$ quantifiers.

Next Time

- ***Functions***

  - How do we model transformations and pairings?

- ***First-Order Definitions***

  - Where does first-order logic come into all of this?

- ***Proofs with Definitions***

  - How does first-order logic interact with proofs?